Appendix 10K

Demonstration that the Per Capita Version of the Quadratic Expenditure Function Can be U-shaped

The estimated expenditure function established using the least squares quadratic regression technique is given by formula [10.32] in Chapter 10, which is reproduced below, with subscripts omitted besides that in EE_{QR}:

$$EE_{QR}(P) = EE_{QR} = A + BP + CP^2 \qquad ...[10K.1]$$

The per capita expenditure function $ee_{QR}(P)$ corresponding to [10K.1] can hence be obtained by dividing [10K.1] throughout by population P, as follows:

$$ee_{QR} = \frac{EE_{QR}}{P} = \frac{A + BP + CP^2}{P} = \frac{A}{P} + B + CP$$

So:

$$ee_{QR} = \frac{A}{P} + B + CP \qquad ...[10K.2]$$

Using differential calculus it can be shown that the first and second derivatives of ee_{QR} with respect to P are as follows:

First derivative: $$\frac{d(ee_{QR})}{dP} = -\frac{A}{P^2} + C = \frac{CP^2 - A}{P^2} = \frac{C(P^2 - \frac{A}{C})}{P^2} \qquad ...[10K.3]$$

Second derivative: $$\frac{d^2(ee_{QR})}{dP^2} = \frac{d}{dP}\left(\frac{d(ee_{QR})}{dP}\right) = \frac{d}{dP}(-\frac{A}{P^2} + C) = \frac{2A}{P^3} \qquad ...[10K.4]$$

It follows from the theory of differential calculus that the graphical representation of the ee_{QR} versus P relationship will be roughly U-shaped (or V-shaped) within the population (P) range of interest, and will have a minimum turning point, if $\frac{d(ee_{QR})}{dP} = 0$ and $\frac{d^2(ee_{QR})}{dP^2} > 0$.

According to expression [10K.3], the first derivative $\frac{d(ee_{QR})}{dP}$ is zero if and only if $(P^2 - \frac{A}{C}) = 0$; that is, when $P^2 = \frac{A}{C}$. So:

$$\frac{d(ee_{QR})}{dP} = 0 \text{ if and only if } P = \pm\sqrt{\frac{A}{C}} \qquad ...[10K.5]$$

But because populations must always be positive numbers, the negative square root can be eliminated, so [10K.5] reduces to:

$$\frac{d(ee_{QR})}{dP} = 0 \text{ if and only if } P = \sqrt{\frac{A}{C}} \qquad ...[10K.6]$$

Now the expression $P = \sqrt{\frac{A}{C}}$ can only have real number values if C is non-zero and $\frac{A}{C} \geq 0$ (in order to avoid having a negative number under the square root sign). It hence follows from [10K.6] that:

$$\frac{d(ee_{QR})}{dP} = 0 \text{ if and only if } C \neq 0,\ \frac{A}{C} \geq 0 \text{ and } P = \sqrt{\frac{A}{C}} \qquad ...[10K.7]$$

So if A is non-zero, then A and C must both be positive or both be negative in order to generate a real-numbered P value at which there will be a turning point where $\frac{dee_{QR}}{dP} = 0$.

According to [10K.4], the second derivative $\frac{d^2(ee_{QR})}{dP^2}$ is positive if A is positive and negative if A is negative, noting that P is always positive. So:

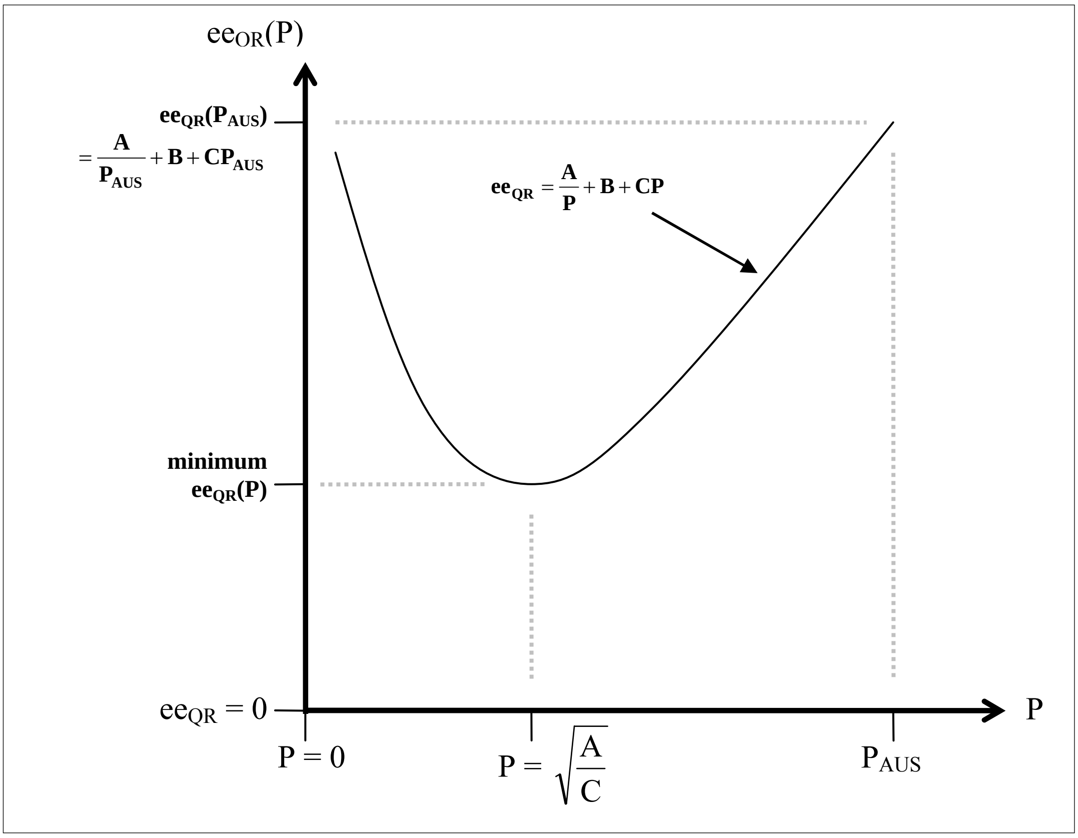
$$\frac{d^2(ee_{QR})}{dP^2} > 0 \text{ if and only if } A > 0 \qquad ...[10K.8]$$

Combining [10K.7] and [10K.8] hence gives that:

$$\frac{d(ee_{QR})}{dP} = 0 \text{ and } \frac{d^2(ee_{QR})}{dP^2} > 0 \text{ if and only if } A > 0,\ C > 0 \text{ and } P = \sqrt{\frac{A}{C}} \qquad ...[10K.9]$$

It follows from [10K.9] that the per capita expenditure versus population (ee_{QR} versus P) relationship will be U-shaped for P values of interest ranging from 0 to P_{AUS}, if A and C are both positive and $P_{AUS} > \sqrt{\frac{A}{C}}$, with the minimum ee_{QR} value occurring at $P = \sqrt{\frac{A}{C}}$, as illustrated in Figure 10K-1 below.

Figure 10-1: U-shaped $ee_{QR}(P)$ curve for A > 0, C > 0 and $P_{AUS} > \sqrt{\frac{A}{C}}$



Of the 253 regression analyses carried out across the 23 CATs and 11 regression sets, the requirements for a U-shaped ee_{QR} curve (A > 0, C > 0 and $P_{AUS} > \sqrt{\frac{A}{C}}$) are fulfilled for 110 (or 43%) of the 253 cases, and an upside down U-shaped ee_{QR} curve arises for a further 43 (17%) of the 253 analyses, for cases where (A < 0, C < 0 and $P_{AUS} > \sqrt{\frac{A}{C}}$), as can be verified using the A and C coefficients presented in Tables 10G-3a and 10G-4a in Appendix 10G. Of these 153 cases in total (the 110 for which U-shaped curves arise, along with the 43 for which upside down U-shaped curves emerge):

- for five cases (SSW for 8ST, 7MA, 7MV and 7MW, and TAC for 7MQ), the minimum expenditure occurs for public sector expenditure categories at populations $P = \sqrt{\frac{A}{C}}$ below the average population of Australia's eight States and Territories ($P_{AUS,4YA} \div 8 =$ 2.3959 million), and hence on this basis support New States or Regional Federation type models with eight or more STTUs; and

- for four cases (BFD for 8ST, 7MA, 7MQ and 5MS), the maximum expenditure occurs for private sector expenditure categories at populations $P = \sqrt{\frac{A}{C}}$ below the average population of Australia's eight States and Territories, and hence support New States or Regional Federation type models with eight or more STTUs.

Besides the nine cases identified above, most of the other 144 cases in which U-shaped curves arise support Fewer States or Regional States type models with eight or less STTUs models (and mostly four or less STTUs), or Unification models. Specifically:

- for 77 cases, the ***minimum*** expenditure occurs for ***public*** sector expenditure categories at populations $P = \sqrt{\frac{A}{C}}$ above the average population of Australia's eight States and Territories ($P_{AUS,4YA} \div 8 = 2.3959$ million), but below Australia's total population ($P_{AUS,4YA} = 19.1669$ million), and hence on this basis support Fewer States or Regional Federation type models with eight or less STTUs (and for 44 of these 77 cases, models with four or less STTUs are supported); and

- for 11 cases (BFD for 8ST, 7MA, 7MQ and 5MS), the ***maximum*** expenditure occurs for ***private*** sector expenditure categories at populations $P = \sqrt{\frac{A}{C}}$ above the average population of Australia's eight States and Territories ($P_{AUS,4YA} \div 8 = 2.3959$ million), but below Australia's total population ($P_{AUS,4YA} = 19.1669$ million), and hence on this basis support Fewer States or Regional Federation type models with eight or less STTUs (and for 8 of these 11 cases, models with four or less STTUs are supported).

For the 100 cases (of the 253 regression analyses) besides those in which U-shaped or upside down U-shaped ee_{QR} curves arise, Unification models are supported in 90 of these cases (these being 11 cases for which A is negative and C positive for private sector CATs, 77 cases for which A is positive and C negative for public sector CATs, and 2 further cases for which A and C are both zero, but the minimum turning point occurs at populations $P = \sqrt{\frac{A}{C}}$ exceeding Australia's total population ($P_{AUS,4YA} = 19.1669$ million). For the remaining 10 of these 100 cases, Unification models are opposed (TAC cases for six regression sets, SSW for 6ST, 6PS and 5MS, and BFD for 6PS). These findings are largely consistent with the Unification model results presented in Chapter 12 which show that the TAC expenditure category is that which consistently achieves the most negative relative benefit estimates, and that the SSW and BFD expenditure categories also generate a significant number of negative relative benefit estimates, whereas most Unification model relative benefit estimates are positive for most of the 23 expenditure categories.

Further research and analysis could clarify the significance of these A and C coefficients and U-shaped per capita expenditure curves, but could not affect the results and conclusions of this thesis to any significant extent. Detailed investigations carried out thus far (see, for example, the sheet titled 'Coefficient Summary' in the Microsoft Excel file 'Full QUAD Summary All 11 All Years 25Nov05', as included in the "Excel Files" folder on this Compact Disk) clearly indicate, as briefly summarised above, that these A and C coefficients most strongly support Unification models and Fewer States and Regional Federation models hosting between two and four STTUs, and are hence very largely consistent with the results presented in Chapters 11 to 13 of this thesis. This obviously *should* be the case, as some of the estimated expenditures and relative benefit estimates presented in Chapter 12 at least are actually based on the A, B and C coefficients derived using the quadratic regression technique. All regression coefficients and all relative benefit estimates shown in Chapters 11 to 13 are ultimately based on the same underlying population and expenditure data as listed in Chapter 8.