Appendix 10J

Detailed Descriptions of Progressive Amalgamation Technique Formulations

This appendix has six sections which support the description of the progressive amalgamation (PA) technique provided in Chapter 10, and which should be read in conjunction with the PA technique sample calculation shown in Appendix 10B in order to gain a full description of how PA technique estimates are generated.

The first section shows how the estimation algorithm (EA) technique has been described previously. The second explains how the PA technique improves on the EA technique in several significant respects. The third provides a graphical explanation of the small STU estimates $A_{TAS,CAT}$, $A_{ACT,CAT}$ and $A_{NT,CAT}$ and specifically shows how the TPS entries in Table 10-9 in Chapter 10 are calculated. The fourth presents describes amalgamation equations and associated population, estimated expenditure and relative expenditure equations. The fifth shows mathematical derivations of results [10.85a] through [10.88a] in Chapter 10. The sixth employs results established in the fifth section to refine the estimated expenditure formulas derived in the fourth section. The seventh section then presents derivations of the formula $RE_{AUS,LR-\{S_b,i,N_b\}} = (1–N_b)A_{i,N_b}$. This formula is a generalised version of result [10.33] in Chapter 10, a proof of which has already been provided in Appendix 10C. This formula is first derived for the specific case of $N_b = 5$, and then for the general case for any applicable N_b value.

The Estimation Algorithm (EA) Technique as Previously Described

Following is an extract from previous work (Drummond 2002: 48-49) which describes the estimation algorithm technique which the progressive amalgamation (PA) technique is designed to improve upon.

> Equation [4] (**C_{DN}[L ≈ 726] ≈ – 7 × FC ≈ – $11.12b**) is accurate to the extent that state and territory FC values are well approximated by [2]. Modest departures from the regression line are evident in Figure 3, however, which could be due to relatively low or high fixed or overhead costs (FC) or relatively low or high marginal per capita costs (MC), or a combination of these. But, whereas marginal per capita costs – of schools, hospitals, teachers, nurses and so on – could be expected to accrue at more or less equal levels in both larger and smaller federal units, fixed or overhead costs can be expected to be higher in centralised political units which govern larger areas and hence need to exercise functional command, control and communication more remotely from communities, through more levels of delegation and with greater coordination

burdens (Oates 1972:35; Boyne 1998:53-54). Accordingly, it shall be assumed that regression line departures are accounted for by higher or lower FC values.

Subsequent estimations rely upon FC estimates for the ACT and Tasmanian public sectors. In percentage terms, the ACT data points deviate most from the regression line (see Figure 4), being on average $1.3453 billion, or 37%, below the line, so, to account for this deviation, the ACT's FC value is reduced this $1.3453b amount below that in [2], to:

$$\mathbf{FC_{ACT} \approx \$0.2430b} \qquad \mathbf{...[5]}$$

Similarly accounting for the Tasmanian data gives:

$$\mathbf{FC_{TAS} \approx \$0.8442b} \qquad \mathbf{...[6]}$$

Such departures can be systematically, if not fully, addressed by an algorithm that estimates the cost savings likely to be achieved through each of the seven horizontal amalgamations of two state-territory units into one that will transform from the present system into the Dual National model.

...

... for N = 8, representing the status quo with the present eight states and territories, the following known 'initial value' applies:

$$\mathbf{C_{NS}[N = 8] = 0} \qquad \mathbf{...[9]}$$

The estimation algorithm employed herein can be described by the following recursion formula which is applied for all N values from 8 down to 2:

$$\mathbf{C_{NS}[N - 1] = C_{NS}[N] - S_A[N]} \qquad \mathbf{...[10]}$$

where N = number of state or territory type government units yet to be amalgamated, such that amalgamations occur in ascending order of *political size*

$S_A[N]$ = estimated savings achieved through the horizontal amalgamation that reduces the number of state-territory type governments from N to (N – 1)

= FC value, being [5] and [6] for N = 8 and 7 respectively, and that obtained in the regression analysis involving the N yet to be amalgamated political units, for N = 6 to N = 2 only, as presented in Table 4.

Results [7], [9] and [10] hence provide that:

$$\mathbf{C_{DN}[L \approx 726] = C_{NS}[N = 1] = - [\, S_A(8) + S_A(7) + S_A(6) + S_A(5) + S_A(4) + S_A(3) + S_A(2) \,]} \qquad \mathbf{...[11]}$$

The algorithm starts with the present eight states and territories and *converges* towards the whole-of-Australia-sized new single state government in the Dual National model, and hence $C_{DN}[L \approx 726]$, one horizontal amalgamation at a time, in ascending order of *political size* (see Table 1). So as N steps down one at a time, the remaining political units increasingly display the large population and land area characteristics of the Australia-wide new single state. For the first two amalgamations, in which the ACT and then TAS would be absorbed into the remaining system, the FC estimates in [5] and [6] are employed:

$$\mathbf{S_A[8] = FC_{ACT} \approx \$0.2430b} \qquad \mathbf{...[12]}$$

and

$$\mathbf{S_A[7] = FC_{TAS} \approx \$0.8442b} \qquad \mathbf{...[13]}$$

For subsequent amalgamations, for N = 6 down to N = 2, results based on the regression analysis of all eight states and territories (such as [12] and [13]) would no longer be adequate, being biased by the ACT and TAS data. Table 3 summarises the regression analyses of the public sector expenditure and population data of the N largest states and territories, in terms of political size (as in Table 1), for N = 6 down to N = 2.

Table 4. Results from Regression Analyses for N = 6 Down to N = 2

Number of Political Units = N	6	5	4	3	2
$S_A[N]$ = FC ($b)	2.5798	3.4805	4.5881	4.6730	3.8085
r^2	0.9896	0.9852	0.9806	0.9979	0.9973

With results [12] and [13] and the $S_A[N]$ figures in the second row of Table 4 above, for N = 6 down to N = 2, [11] becomes:[6]

$$\mathbf{C_{DN}[L \approx 726] = C_{NS}[N = 1] \approx -\$20.22b} \qquad \text{...[14]}$$

Result [14] estimates the savings achievable *at the state and territory level* through the elimination of horizontally duplicated fixed or overhead costs among state and territory governments, but further *coordination cost* savings are likely to be achieved by the new single state government and also by the federal government.

The PA Technique's Improvements on the EA Technique

The progressive amalgamation (PA) techniques described in Chapter 10 and in this appendix have been designed to improve on the estimation algorithm (EA) technique described above, in several respects, as follows:

- As with all RB estimation techniques described in Chapter 10, the PA technique is designed to generate RE and RB estimates which are meaningful and applicable across public and private sector expenditure categories alike, whereas the EA technique calculated relative financial benefits in terms of cost savings only and was hence only applicable to public sector CATs only.
- The PA technique considers a range of different amalgamation sequences, which refer to the order in which the STUs amalgamate. Five of these amalgamation sequences, among the numerous possibilities, are employed in this study and described further below. The PA technique also establishes four separate RB estimates for each amalgamation sequence. The EA technique only obtained a single estimate for a single amalgamation sequence.
- The EA and PA techniques alike involve two amalgamation stages: an initial stage in which smaller STUs are absorbed into larger ones, and a second stage involving amalgamation steps which converge, one STTU at a time, towards the Australia-wide single STTU (AWSS) of the DNC model. In the PA technique, it is assumed that TAS, ACT and NT are amalgamated with (or absorbed into) VIC, NSW and SA respectively in the initial amalgamation stage, whereas the corresponding stage in the EA technique only involved the absorption of the ACT into NSW and of TAS into VIC. So whereas the EA technique singled out the ACT and TAS from the other STUs on the basis of being too small in population and land area to meaningfully reflect the Australia-wide scale of the DNC and NCL type models, for which these estimation techniques are designed, the PA technique adds the NT to this set of STUs.

- The EA technique assumes that marginal per capita costs are more or less the same for the ACT, TAS and all other STUs, and that expenditure deviations above or below linear regression lines are explained by variations in fixed or overhead costs – FC_{ACT} and FC_{TAS} for the ACT and TAS respectively – rather than by variations in marginal per capita costs. The PA technique assumes similarly, but (1) considers the NT along with the ACT and TAS as above, (2) refers to fixed or overhead *expenditures* (given by A_{LR} in [10.30b]) and marginal per capita *expenditures* (B_{LR} in [10.30b]), rather than *costs*, and (3) addresses a possible flaw in the EA approach (though one that had very minimal impact on estimates presented in 2002). In the EA technique, the fixed or overhead costs for the ACT and TAS were calculated in relation to the regression equation established for all eight STUs (thus the 8ST regression set as classified here). But this approach can be considered suspect as the ACT and TAS contributed to the very regression equation from which their deviation was supposed to be calculated, so the FC_{ACT} and FC_{TAS} values do not reflect the true deviation of the ACT and TAS from the general pattern evident among the *other* STUs only.

Graphical Explanation of Small STU Estimates $A_{TAS,CAT}$, $A_{ACT,CAT}$ and $A_{NT,CAT}$

Figures 10J-1a and 10J-1b below graphically illustrate how the small STU estimates $A_{TAS,CAT}$, $A_{ACT,CAT}$ and $A_{NT,CAT}$ are established, and specifically demonstrates how the TPS figures in Table 10-9 in Chapter 10 are calculated.

The lines shown for TAS, ACT and NT in Figure 10J-1a have the same gradient as the 5MS LR line shown, but are vertically shifted downward and hence intercept the vertical axis below the vertical axis intercept of the 5MS extension. Figure 10J-1b provides a close-up view of the TAS, ACT and NT lines and the 5MS extension line, and provides the specific intercept values (in $b).

Figure 10J-1a: Graphical Description of Small STU Expenditure Lines Assuming Same Gradient as 5MS Line

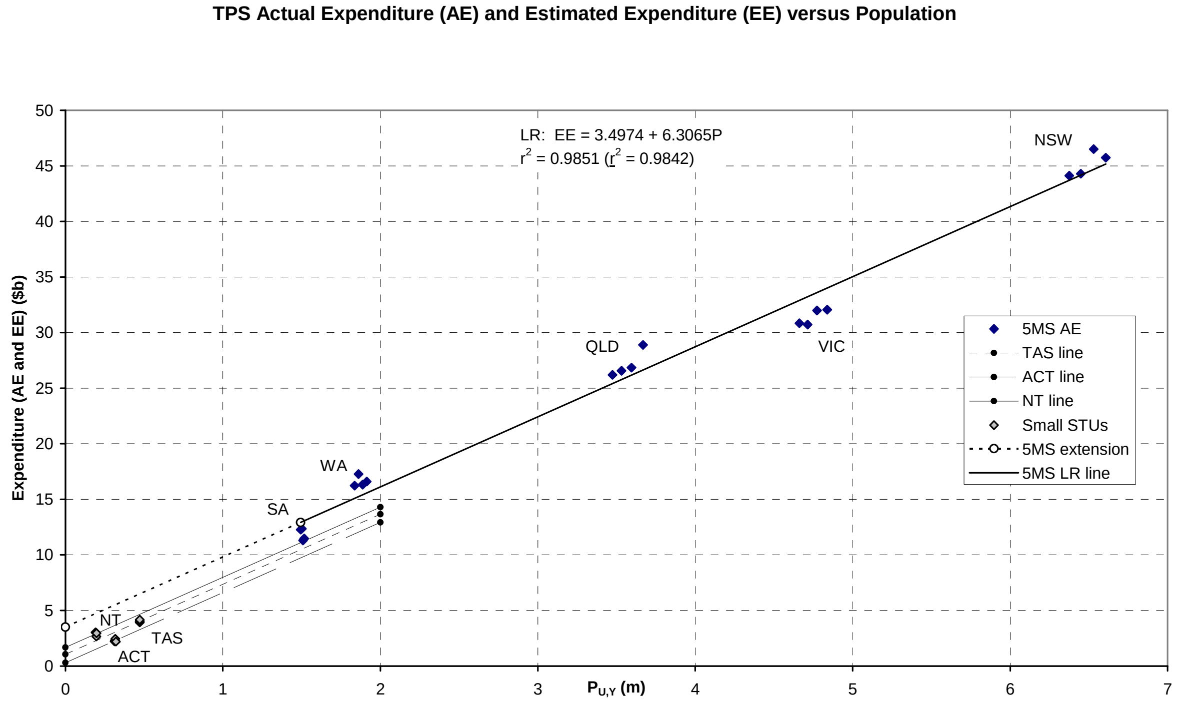
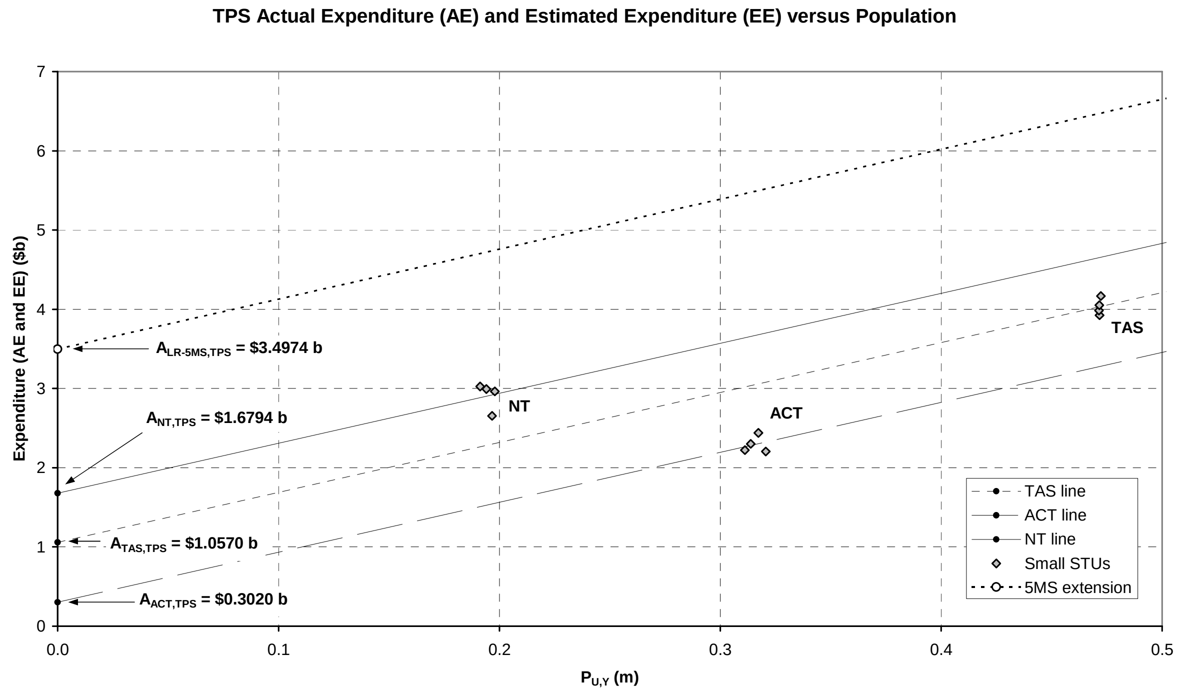


Figure 10J-1b: Graphical Description of Small STU Estimates $A_{TAS,TPS}$, $A_{ACT,TPS}$ and $A_{NT,TPS}$



Amalgamation Equations and Associated Population, Estimated Expenditure and Relative Expenditure Equations

This section formally defines amalgamation equations as shown in Table 10-8 in Chapter 10 and derives associated population, estimated expenditure and relative expenditure equations which describe an amalgamation step.

General Form of Amalgamation Equation

The following equation describes the sequence i amalgamation of the STTU pair $U_{i,N_b[k_1]}$ and $U_{i,N_b[k_2]}$ which produces the consolidated unit $U_{i,N_b[k_1]}$-$U_{i,N_b[k_2]}$:

$$U_{i,N_b[k_1]} + U_{i,N_b[k_2]} \rightarrow U_{i,N_b[k_1]}\text{-}U_{i,N_b[k_2]} \qquad \text{...[10J.1a]}$$

or, in a simplified notation, for any two STTUs $U_{i,\alpha}$ and $U_{i,\beta}$, assuming that N_b, k_1 and k_2 are known:

$$U_{i,\alpha} + U_{i,\beta} \rightarrow U_{i,\alpha}\text{-}U_{i,\beta} \qquad \text{...[10J.1b]}$$

The amalgamation given by [10J.1b] can be described as $Am_{i,\{U_{i,\alpha}, U_{i,\beta}\},\{U_{i,\alpha}\text{-}U_{i,\beta}\}}$ in amalgamation notation, or by the alternative notation $Am_{i[U_{i,\alpha} + U_{i,\beta} \rightarrow U_{i,\alpha}\text{-}U_{i,\beta}]}$. The amalgamation equations shown in Table 10-8 in Chapter 10 provide specific examples of the equations given in general form by [10J.1a] and [10J.1b].

Population of an Amalgamated STTU

The population of the amalgamated STTU formed as in [10J.1b] is hence given by:

$$P_{U_{i,\alpha}\text{-}U_{i,\beta}} = P_{U_{i,\alpha}} + P_{U_{i,\beta}} \qquad \text{...[10J.2]}$$

Relative Expenditure of an Amalgamated STTU

The relative expenditure $RE_{U_{i,\alpha}\text{-}U_{i,\beta}}$ of an amalgamated STTU $U_{i,\alpha}$-$U_{i,\beta}$ is defined as the estimated expenditure $EE_{U_{i,\alpha}\text{-}U_{i,\beta}}$ of $U_{i,\alpha}$-$U_{i,\beta}$ relative to the sum of the estimated expenditures $EE_{U_{i,\alpha}}$ and $EE_{U_{i,\beta}}$ of STTUs $U_{i,\alpha}$ and $U_{i,\beta}$ respectively prior to amalgamation, as follows (where estimated expenditures EE will be known actual expenditures (AE) in cases where either or both of $U_{i,\alpha}$ or $U_{i,\beta}$ are one or two of the eight original STUs prior to amalgamation):

$$RE_{U_{i,\alpha}\text{-}U_{i,\beta}} = EE_{U_{i,\alpha}\text{-}U_{i,\beta}} - (EE_{U_{i,\alpha}} + EE_{U_{i,\beta}}) \qquad \text{...[10J.3]}$$

Formula [10J.3] can also be expressed in terms of estimated expenditure:

$$EE_{U_{i,\alpha}\text{-}U_{i,\beta}} = (EE_{i,\alpha} + EE_{i,\beta}) + RE_{U_{i,\alpha}\text{-}U_{i,\beta}} \quad ...[10J.4]$$

As described in Chapter 10, the PA technique employs linear regression methods for amalgamations involving five and fewer STTUs. The linear equation for the regression involving N_b STTUs arising in sequence i is hence given by:

$$EE_{i,N_b} = A_{i,N_b} + B_{i,N_b}P \text{ for } 2 \le N_b \le 5 \quad ...[10J.5a]$$

So for N_b =5 down to 2:

$$EE_{i,5} = A_{i,5} + B_{i,5}P \quad \text{for } N_b = 5 \quad ...[10J.5b]$$

$$EE_{i,4} = A_{i,4} + B_{i,4}P \quad \text{for } N_b = 4 \quad ...[10J.5c]$$

$$EE_{i,3} = A_{i,3} + B_{i,3}P \quad \text{for } N_b = 3 \quad ...[10J.5d]$$

and

$$EE_{i,2} = A_{i,2} + B_{i,2}P \quad \text{for } N_b = 2 \quad ...[10J.5e]$$

So with [10J.5e] applying for the case of N_b = 2:

$$EE_{U_{i,2[1]}} = A_{i,2} + B_{i,2}P_{U_{i,2[1]}} \quad ...[10J.6a]$$

$$EE_{U_{i,2[2]}} = A_{i,2} + B_{i,2}P_{U_{i,2[2]}} \quad ...[10J.6b]$$

and

$$EE_{U_{i,2[1]}\text{-}U_{i,2[2]}} = A_{i,2} + B_{i,2}(P_{U_{i,2[1]}} + P_{U_{i,2[2]}}) \quad ...[10J.6c]$$

Derivation of Formulas [10.85a] to [10.88a] in Chapter 10

This section derives formulas [10.88a] through [10.85a] in turn below, beginning with [10.88a] for N_b = 2.

Derivation of Formula [10.88a] for N_b = 2

Formula [10.88a] quantifies the relative expenditure $RE_{i,2,1} = RE_{U_{i,2[1]}\text{-}U_{i,2[2]}}$ for the sequence i amalgamation step that reduces the number of STTUs from 2 to 1.

Substituting [10J.6a], [10J.6b] and [10J.6c] into [10J.3] gives that:

$$\begin{aligned} RE_{i,2,1} &= RE_{U_{i,2[1]}\text{-}U_{i,2[2]}} \\ &= A_{i,2} + B_{i,2}(P_{U_{i,2[1]}} + P_{U_{i,2[2]}}) - (A_{i,2} + B_{i,2}P_{U_{i,2[1]}} + A_{i,2} + B_{i,2}P_{U_{i,2[2]}}) \end{aligned}$$

which, on cancellation, reduces to

$RE_{i,2,1} = RE_{U_{i,2[1]}\text{-}U_{i,2[2]}} = -A_{i,2}$ ...[10J.7]

which further reduces to formula [10.88a] as follows:

$RE_{i,2,1} = -A_{i,2}$ ...[10.88a]

Derivation of Formula [10.87a] for N_b = 3

Formula [10.87a] quantifies the relative expenditure $RE_{i,3,2} = RE_{U_{i,3[1]}\text{-}U_{i,3[2]}}$ for the sequence i amalgamation step that reduces the number of STTUs from 3 to 2.

For the case of $N_b = 3$, [10J.1a] provides that the amalgamation of two of the three STTUs is given by [10J.8] below, where it is assumed that $U_{i,3[1]}$ and $U_{i,3[2]}$ are the amalgamating STTU pair:

$U_{i,3[1]} + U_{i,3[2]} \rightarrow U_{i,3[1]}\text{-}U_{i,3[2]}$ ...[10J.8]

Formula [10J.5d] also gives that:

$EE_{U_{i,3[1]}} = A_{i,3} + B_{i,3}P_{U_{i,3[1]}}$ ...[10J.9a]

$EE_{U_{i,3[2]}} = A_{i,3} + B_{i,3}P_{U_{i,3[2]}}$ ...[10J.9b]

$EE_{U_{i,3[3]}} = A_{i,3} + B_{i,3}P_{U_{i,3[3]}}$ ...[10J.9c]

and

$EE_{U_{i,3[1]}\text{-}U_{i,3[2]}} = A_{i,3} + B_{i,3}(P_{U_{i,3[1]}} + P_{U_{i,3[2]}})$ ...[10J.9d]

Substituting [10J.9a], [10J.9b] and [10J.9d] into [10J.3] hence gives that:

$RE_{i,3,2} = RE_{U_{i,3[1]}\text{-}U_{i,3[2]}}$

$= A_{i,3} + B_{i,3}(P_{U_{i,3[1]}} + P_{U_{i,3[2]}}) - (A_{i,3} + B_{i,3}P_{U_{i,3[1]}} + A_{i,3} + B_{i,3}P_{U_{i,3[2]}})$ ...[10J.10]

which reduces to

$RE_{i,3,2} = RE_{U_{i,3[1]}\text{-}U_{i,3[2]}} = -A_{i,3}$ ...[10J.11]

and further reduces to formula [10.87a] as follows:

$RE_{i,3,2} = -A_{i,3}$ ...[10.87a]

The $U_{i,3[3]}$ estimated expenditure formula [10J.9c] is not substituted into [10J.3] along with [10J.8a] and [10J.8b], to give [10J.10] and then [10.87a] above, because the amalgamation given by [10J.8] involves STTUs $U_{i,3[1]}$ and $U_{i,3[2]}$ only and leaves the third STTU $U_{i,3[3]}$ unchanged in terms of both population and expenditure.

Derivation of Formula [10.86a] for N_b = 4

Formula [10.86a] quantifies the relative expenditure $RE_{i,4,3} = RE_{U_{i,4[1]}\text{-}U_{i,4[2]}}$ for the sequence i amalgamation step that reduces the number of STTUs from 4 to 3.

For the case of $N_b = 4$, the amalgamation of two of the four STTUs is given by [10J.1a] as follows where it is assumed that $U_{i,4[1]}$ and $U_{i,4[2]}$ are the amalgamating STTU pair:

$$U_{i,4[1]} + U_{i,4[2]} \rightarrow U_{i,4[1]}\text{-}U_{i,4[2]} \qquad \text{...[10J.12]}$$

Formula [10J.5c] also gives that:

$$EE_{U_{i,4[1]}} = A_{i,4} + B_{i,4}P_{U_{i,4[1]}} \qquad \text{...[10J.13a]}$$

$$EE_{U_{i,4[2]}} = A_{i,4} + B_{i,4}P_{U_{i,4[2]}} \qquad \text{...[10J.13b]}$$

$$EE_{U_{i,4[3]}} = A_{i,4} + B_{i,4}P_{U_{i,4[3]}} \qquad \text{...[10J.13c]}$$

$$EE_{U_{i,4[4]}} = A_{i,4} + B_{i,4}P_{U_{i,4[3]}} \qquad \text{...[10J.13d]}$$

and

$$EE_{U_{i,4[1]}\text{-}U_{i,4[2]}} = A_{i,4} + B_{i,4}(P_{U_{i,4[1]}} + P_{U_{i,4[2]}}) \qquad \text{...[10J.13e]}$$

Substituting [10J.13a], [10J.13b] and [10J.13e] into [10J.3] hence gives that:

$$RE_{i,4,3} = RE_{U_{i,4[1]}\text{-}U_{i,4[2]}}$$
$$= A_{i,4} + B_{i,4}(P_{U_{i,4[1]}} + P_{U_{i,4[2]}}) - (A_{i,4} + B_{i,4}P_{U_{i,4[1]}} + A_{i,4} + B_{i,4}P_{U_{i,4[2]}}) \qquad \text{...[10J.14]}$$

which reduces to

$$RE_{i,4,3} = RE_{U_{i,4[1]}\text{-}U_{i,4[2]}} = -A_{i,4} \qquad \text{...[10J.15]}$$

and further reduces to formula [10.86a] as follows:

$$RE_{i,4,3} = -A_{i,4} \qquad \text{...[10.86a]}$$

The $U_{i,4[3]}$ and $U_{i,4[4]}$ estimated expenditure formulas [10J.13c] and [10J.13d] are not substituted into [10J.3] along with [10J.13a] and [10J.13b], to give [10J.14] and [10.86a] above, because the amalgamation given by [10J.12] involves STTUs $U_{i,4[1]}$ and $U_{i,4[2]}$ only and leaves $U_{i,4[3]}$ and $U_{i,4[4]}$ unchanged in terms of both population and expenditure.

Derivation of Formula [10.85a] for N_b = 5

Formula [10.85a] quantifies the relative expenditure $RE_{i,5,4} = RE_{U_{i,5[1]}\text{-}U_{i,5[2]}}$ for the sequence i amalgamation step that reduces the number of STTUs from 5 to 4.

For the case of $N_b = 5$, the amalgamation of two of the five STTUs is given by [10J.1a] as follows where it is assumed that $U_{i,5[1]}$ and $U_{i,5[2]}$ are the amalgamating STTU pair:

$$U_{i,5[1]} + U_{i,5[2]} \rightarrow U_{i,5[1]}\text{-}U_{i,5[2]} \qquad \text{...[10J.16]}$$

Formula [10J.5b] also gives that:

$$EE_{U_{i,5[1]}} = A_{i,5} + B_{i,5}P_{U_{i,5[1]}} \qquad \text{...[10J.17a]}$$

$$EE_{U_{i,5[2]}} = A_{i,5} + B_{i,5}P_{U_{i,5[2]}} \qquad \text{...[10J.17b]}$$

$$EE_{U_{i,5[3]}} = A_{i,5} + B_{i,5}P_{U_{i,5[3]}} \qquad \text{...[10J.17c]}$$

$$EE_{U_{i,5[4]}} = A_{i,5} + B_{i,5}P_{U_{i,5[3]}} \qquad \text{...[10J.17d]}$$

$$EE_{U_{i,5[5]}} = A_{i,5} + B_{i,5}P_{U_{i,5[3]}} \qquad \text{...[10J.17e]}$$

and

$$EE_{U_{i,5[1]}\text{-}U_{i,5[2]}} = A_{i,5} + B_{i,5}(P_{U_{i,5[1]}} + P_{U_{i,5[2]}}) \qquad \text{...[10J.17f]}$$

Substituting [10J.17a], [10J.17b] and [10J.17f] into [10J.3] hence gives that:

$$RE_{i,5,4} = RE_{U_{i,5[1]}\text{-}U_{i,5[2]}}$$
$$= A_{i,5} + B_{i,5}(P_{U_{i,5[1]}} + P_{U_{i,5[2]}}) - (A_{i,5} + B_{i,5}P_{U_{i,5[1]}} + A_{i,5} + B_{i,5}P_{U_{i,5[2]}}) \qquad \text{...[10J.18]}$$

which reduces to

$$RE_{i,5,4} = RE_{U_{i,5[1]}\text{-}U_{i,5[2]}} = -A_{i,4} \qquad \text{...[10J.19]}$$

and further reduces to formula [10.85a] as follows:

$$RE_{i,5,4} = -A_{i,5} \qquad \text{...[10.85a]}$$

The $U_{i,5[3]}$, $U_{i,5[4]}$ and $U_{i,5[5]}$ estimated expenditure formulas [10J.17c], [10J.17d] and [10J.17e] are not substituted into [10J.3] along with [10J.17a] and [10J.17b], to give [10J.18] and [10.85a] above, because the amalgamation given by [10J.16] involves STTUs $U_{i,5[1]}$ and $U_{i,5[2]}$ only and leaves $U_{i,5[3]}$, $U_{i,5[4]}$ and $U_{i,5[5]}$ unchanged in terms of both population and expenditure.

Generalised Version of Formulas [10.85a] to [10.88a]

Formulas [10.85a] through [10.88a], viewed together, show that the relative expenditure for the sequence i amalgamation step which reduces the number of STTUs from N_b to N_b–1 is given by:

$$RE_{i,N_b,N_b-1} = RE(Am_{i,N_b,\,N_b-1}) = -A_{i,N_b} \qquad \text{for } 2 \le N_b \le 5 \qquad ...[10J.20]$$

Further Refinement of Estimated Expenditure Formulas

For the case of $EE_{i,\alpha} = EE_{U_{i,2[1]}}$ and $EE_{i,\beta} = E_{U_{i,2[2]}}$, [10J.4] above becomes:

$$EE_{U_{i,2[1]}\text{-}U_{i,2[2]}} = (EE_{U_{i,2[1]}} + EE_{U_{i,2[2]}}) + RE_{U_{i,2[1]}\text{-}U_{i,2[2]}} \qquad ...[10J.21]$$

which becomes as follows noting [10J.7]

$$EE_{U_{i,2[1]}\text{-}U_{i,2[2]}} = (EE_{U_{i,2[1]}} + EE_{U_{i,2[2]}}) - A_{i,2} \qquad ...[10J.22]$$

It can similarly be shown that the following formula applies for the i sequence amalgamation between STTUs $U_{i,N_b[1]}$ and $U_{i,N_b[2]}$ for all N_b in the range $2 \le N_b \le 5$:

$$EE_{U_{i,N_b[1]}\text{-}U_{i,N_b[2]}} = (EE_{U_{i,N_b[1]}} + EE_{U_{i,N_b[2]}}) - A_{i,N_b} \qquad ...[10J.23a]$$

So for N_b = 3, 4 and 5, [10J.23] gives that:

$$EE_{U_{i,3[1]}\text{-}U_{i,3[2]}} = (EE_{U_{i,3[1]}} + EE_{U_{i,3[2]}}) - A_{i,3} \qquad ...[10J.23b]$$

$$EE_{U_{i,4[1]}\text{-}U_{i,4[2]}} = (EE_{U_{i,4[1]}} + EE_{U_{i,4[2]}}) - A_{i,4} \qquad ...[10J.23c]$$

and

$$EE_{U_{i,5[1]}\text{-}U_{i,5[2]}} = (EE_{U_{i,5[1]}} + EE_{U_{i,5[2]}}) - A_{i,5} \qquad ...[10J.23d]$$

Result [10J.23a] above is also applied to N_b = 6, 7 and 8 as follows for the purposes of establishing estimated expenditures for the amalgamated units NSW-ACT, VIC-TAS and SA-NT:

$$EE_{U_{i,6[1]}\text{-}U_{i,6[2]}} = (EE_{U_{i,6[1]}} + EE_{U_{i,6[2]}}) - A_{i,6} \qquad ...[10J.23e]$$

$$EE_{U_{i,7[1]}\text{-}U_{i,7[2]}} = (EE_{U_{i,7[1]}} + EE_{U_{i,7[2]}}) - A_{i,7} \qquad ...[10J.23f]$$

and

$$EE_{U_{i,8[1]}\text{-}U_{i,8[2]}} = (EE_{U_{i,8[1]}} + EE_{U_{i,8[2]}}) - A_{i,8} \qquad ...[10J.23g]$$

The units $U_{i,6[1]}$, $U_{i,6[2]}$, $U_{i,7[1]}$, $U_{i,7[2]}$, $U_{i,8[1]}$, and $U_{i,8[2]}$ in [10J.23e] through [10J.23g] above are NSW, ACT, VIC, TAS, SA and NT in the order in the order determined by the given amalgamation sequence as shown in Table 10-8 of Chapter 10. The $A_{i,6}$, $A_{i,7}$ and $A_{i,8}$ values in [10J.23e] through [10J.23g], similarly, are the A_{TAS}, A_{ACT} and A_{NT} values shown in Table 10-9 of Chapter 10. Formulas [10J.23e] through [10J.23g] can hence be written as formulas [10.77] through [10.79] in Chapter 10:

$$EE_{NSW\text{-}ACT} = (AE_{NSW} + AE_{ACT}) - A_{ACT} \quad \text{...[10.77]}$$

$$EE_{VIC\text{-}TAS} = (AE_{VIC} + AE_{TAS}) - A_{TAS} \quad \text{...[10.78]}$$

and

$$EE_{SA\text{-}NT} = (AE_{SA} + AE_{NT}) - A_{NT} \quad \text{...[10.79]}$$

Derivation of the Formula $RE_{AUS,LR\text{-}\{S_b,i,N_b\}} = (1–N_b)A_{i,N_b}$

Appendix 10C provided a mathematical derivation of formula [10.33] in Chapter 10, as follows:

$$RE_{AUS,LR\text{-}8ST} = -7A_{LR} \quad \text{...[10.33]}$$

As noted in Chapter 10, and explained in detail in Appendix 10C, formula [10.33] applies to the LR technique when the 8ST regression set is used because the populations of the eight STUs of the 8ST regression set add to the total Australian population P_{AUS}, and mathematical results hence simplify to [10.33], but no formula similar to [10.33] can be derived for any of the 10 regression sets other than 8ST because the sum total populations of the STUs which make up these other REG sets in all cases fall below P_{AUS}, so the simplifications that generate [10.33] for 8ST do not occur for any other REG sets. A generalised version of [10.33] can, however, be established, for the STTU sets $\{\mathbf{S}_{b,i,N_b}\}$ that arise in the PA technique amalgamation processes, as defined in Chapter 10, because the sum total populations of the STTUs of these STTU sets in all cases add to P_{AUS}. This section provides a mathematical proof of the following result, which can be considered a generalised version of [10.33] in Chapter 10 which applies to STTU sets comprising N_b STTUs, where N_b is a counting number between 2 and 8:

$$RE_{AUS,LR\text{-}\{\mathbf{S}_b,i,N_b\}} = (1–N_b)A_{i,N_b} \quad \text{...[10J.24]}$$

Three of the four PA technique estimates, for j = 2, 3 and 4, involve an application of the LR technique to STTU sets that arise in the five amalgamation sequences considered. For j = 2, the LR technique is applied to the STTU set comprising three STTUs. And for j = 3 and j = 4 respectively, the LR technique is applied to the STTU set containing four and five STTUs.

Result [10J.24] is first proven for the case $N_b = 5$ involving the five STTUs that arise in the PA technique following the amalgamations which form NSW-ACT, VIC-TAS and SA-NT. A general proof applicable for any N_b then follows.

Derivation of the Formula $RE_{AUS,LR\text{-}\{S_b,i,5\}} = -4A_{i,5}$ for the LR Technique Applied to the Five STTUs Resulting from the Formation of NSW-ACT, VIC-TAS and SA-NT

Result [10.60e] in Chapter 10 provides that the set of five STTUs which results after the first three STTU amalgamation steps is given by:

$$\{\mathbf{S}_{b,i,5}\} = \{U_{i,5[1]}, U_{i,5[2]}, U_{i,5[3]}, U_{i,5[4]}, U_{i,5[5]}\} \qquad ...[10.60e]$$

From Table 10-8 in Chapter 10 it is seen that for all five amalgamation sequences considered in this study:

$$\{\mathbf{S}_{b,i,5}\} = \{U_{i,5[1]}, U_{i,5[2]}, U_{i,5[3]}, U_{i,5[4]}, U_{i,5[5]}\}$$
$$= \{\text{NSW-ACT, VIC-TAS, QLD, WA, SA-NT}\} \qquad ...[10J.25]$$

Result [10.32] also provides that:

$$RE_{AUS,LR} = EE_{AUS,LR} - AE_{AUS} = (A_{LR} + B_{LR}P_{AUS}) - AE_{AUS} \qquad ...[10.32]$$

So if the LR technique is applied in the PA technique at the stage where $N_b = 5$, [10.32] can be written as:

$$RE_{AUS,LR\text{-}\{\mathbf{S}_b,i,5\}} = EE_{AUS,LR\text{-}\{\mathbf{S}_b,i,5\}} - AE_{AUS}$$
$$= (A_{i,5} + B_{i,5}P_{AUS}) - AE_{AUS} \qquad ...[10J.26]$$

And for the hypothetical government structure with the STTU set of [10J.25]:

$$AE_{AUS} = AE_{NSW\text{-}ACT} + AE_{VIC\text{-}TAS} + AE_{QLD} + AE_{WA} + AE_{SA\text{-}NT} \qquad ...[10J.27]$$

So if actual STTU expenditures can be well approximated by the linear regression formula [10.30b] in Chapter 10, as assumed in the LR and PA techniques, then the following formula will apply to the five STTUs in [10J.25] above which result from the first three amalgamation steps of the PA technique:

$$AE_U = EE_U = A_{i,5} + B_{i,5}P_U \qquad ...[10J.28a]$$

The following results would therefore apply for the five STTUs defined in [10J.25]:

$$AE_{NSW\text{-}ACT} = A_{i,5} + B_{i,5}P_{NSW\text{-}ACT} \qquad ...[10J.28b]$$

$$AE_{VIC\text{-}TAS} = A_{i,5} + B_{i,5}P_{VIC\text{-}TAS} \qquad ...[10J.28c]$$

$$AE_{QLD} = A_{i,5} + B_{i,5}P_{QLD} \qquad ...[10J.28d]$$

$$AE_{WA} = A_{i,5} + B_{i,5}P_{WA} \qquad ...[10J.28e]$$

$$AE_{SA\text{-}NT} = A_{i,5} + B_{i,5}P_{SA\text{-}NT} \qquad ...[10J.28f]$$

Substituting [10J.28b] through [10J.28f] into [10J.27] hence gives:

$$AE_{AUS} = (A_{i,5} + B_{i,5}P_{NSW\text{-}ACT}) + (A_{i,5} + B_{i,5}P_{VIC\text{-}TAS}) + (A_{i,5} + B_{i,5}P_{QLD}) + (A_{i,5} + B_{i,5}P_{WA}) + (A_{i,5} + B_{i,5}P_{SA\text{-}NT})$$

$$= (A_{i,5} + A_{i,5} + A_{i,5} + A_{i,5} + A_{i,5}) + (B_{i,5}P_{NSW\text{-}ACT} + B_{i,5}P_{VIC\text{-}TAS} + B_{i,5}P_{QLD} + B_{i,5}P_{WA} + B_{i,5}P_{SA\text{-}NT})$$

$$= 5A_{i,5} + B_{i,5}(P_{NSW\text{-}ACT} + P_{VIC\text{-}TAS} + P_{QLD} + P_{WA} + P_{SA\text{-}NT})$$

so

$$AE_{AUS} = 5A_{i,5} + B_{i,5}(P_{NSW\text{-}ACT} + P_{VIC\text{-}TAS} + P_{QLD} + P_{WA} + P_{SA\text{-}NT}) \qquad ...[10J.29]$$

But the population figures in brackets in [10J.29] add up to the Australia-wide total population P_{AUS}, as follows:

$$P_{AUS} = P_{NSW\text{-}ACT} + P_{VIC\text{-}TAS} + P_{QLD} + P_{WA} + P_{SA\text{-}NT} \qquad ...[10J.30]$$

So with [10J.30], [10J.29] simplifies to

$$AE_{AUS} = 5A_{i,5} + B_{i,5}P_{AUS} \qquad ...[10J.31]$$

So with [10J.31], [10J.26] becomes:

$$RE_{AUS,LR\text{-}\{s_b,i,5\}} = (A_{i,5} + B_{i,5}P_{AUS}) - (5A_{i,5} + B_{i,5}P_{AUS}) \qquad ...[10J.32]$$

which reduces, on cancellation, to:

$$RE_{AUS,LR\text{-}\{s_b,i,5\}} = -4A_{i,5} \qquad ...[10J.33]$$

Formula [10J.33], which is the same as [10J.24] for the case of $N_b = 5$, provides the basis for the large STTU component of the j = 4 PA technique estimate defined in Chapter 10 by the equivalent results [10.65] and [10.92], as follows:

$$RE_{i(4)} = RE_{i,8,5} + [4RE_{i,5,4}] \qquad ...[10.65]$$

and

$$RE_{i(4)} = -(A_{ACT} + A_{TAS} + A_{NT}) - (4A_{i,5}) \qquad ...[10.92]$$

The large STTU component ($-4A_{i,5}$) in [10.92] is equivalent to the $RE_{AUS,LR\text{-}8ST}$ estimate given by [10.33] above and in Chapter 10. For public sector expenditure categories, this ($-4A_{i,5}$) term can be interpreted as the four multiples of fixed or overhead expenditure (FOE) which could be saved if the five STTUs arising in the PA technique, after the small STU absorptions of first three amalgamation steps, horizontally amalgamated into the single Australia-wide single STTU (AWSS). Prior to this amalgamation process, Australia-wide expenditures would include five multiples of such FOE, and after amalgamation just a single FOE quantum would be incurred, so four lots of FOE, amounting to $4A_{i,5}$ as in [10J.33] and [10.92] above, could be saved. As explained in Appendix 10F, for private sector CATs this ($-4A_{i,5}$) term, like the ($-7A_{LR}$) term in [10.33], can be interpreted as an aggregated fixed cost hurdle faced by private sector business units within STTUs and STTU private sectors as wholes in their quest to achieve financial viability and prosperity.

Derivation of the Formula $RE_{AUS} = (1–N_b)A_{i,N_b}$ for the General Case of N_b STTUs Resulting from the First (8 – N_b) Amalgamation Steps

Result [10.60a] provides that the set of N_b STTUs which results after the first ($8–N_b$) STTU amalgamation steps is as follows, where N_b can range from 8 down to 1, or from 8 down to 2 for non-trivial cases in which at least one amalgamation step remains before the Australia-wide single STTU (AWSS) is reached:

$$\{\mathbf{S}_{b,i,N_b}\} = \{U_{i,N_b[1]}, U_{i,N_b[2]}, ..., U_{i,N_b[N_b]}\} \qquad ...[10.60a]$$

So if the LR technique is applied in the PA technique at the stage where there are N_b STTUs, [10.32] can be written as:

$$RE_{AUS,LR\text{-}\{\mathbf{S}_{b,i,N_b}\}} = EE_{AUS,LR\text{-}\{\mathbf{S}_{b,i,N_b}\}} - AE_{AUS}$$
$$= (A_{i,N_b} + B_{i,N_b}P_{AUS}) - AE_{AUS} \qquad ...[10J.34]$$

And for the hypothetical government structure with the STTU set of [10.60a]:

$$AE_{AUS} = AE_{U_{i,N_b[1]}} + AE_{U_{i,N_b[2]}} + ... + AE_{U_{i,N_b[N_b]}} \qquad ...[10J.35]$$

So if actual STTU expenditures can be well approximated by the linear regression formula [10.30b] in Chapter 10, as assumed in the LR and PA techniques, then the following formula will apply to the N_b STTUs in [10.60a] above which result from the first ($8–N_b$) amalgamation steps of the PA technique:

$$AE_U = EE_U = A_{i,N_b} + B_{i,N_b}P_U \qquad ...[10J.36a]$$

The following results would therefore apply for the N_b STTUs defined in [10.60a]:

$$AE_{U_{i,N_b[1]}} = A_{i,N_b} + B_{i,N_b}\, P_{U_{i,N_b[1]}} \qquad ...[10J.36b]$$

$$AE_{U_{i,N_b[2]}} = A_{i,N_b} + B_{i,N_b}P_{U_{i,N_b[2]}} \qquad ...[10J.36c]$$

$$\vdots$$

$$AE_{U_{i,N_b[N_b]}} = A_{i,N_b} + B_{i,N_b}P_{U_{i,N_b[N_b]}} \qquad ...[10J.36d]$$

Substituting [10J.36b] through [10J.36d] into [10J.35] hence gives:

$$AE_{AUS} = (A_{i,N_b} + B_{i,N_b}P_{U_{i,N_b[1]}}) + (A_{i,N_b} + B_{i,N_b}P_{U_{i,N_b[2]}}) + ... + (A_{i,N_b} + B_{i,N_b}P_{U_{i,N_b[N_b]}})$$

$$= (N_b \times A_{i,N_b}) + (B_{i,N_b}P_{U_{i,N_b[1]}} + B_{i,N_b}P_{U_{i,N_b[2]}} + ...\ B_{i,N_b}P_{U_{i,N_b[N_b]}})$$

$$= N_bA_{i,N_b} + B_{i,N_b}(P_{U_{i,N_b[1]}} + P_{U_{i,N_b[2]}} + ... + P_{U_{i,N_b[N_b]}})$$

so

$$AE_{AUS} = N_bA_{i,N_b} + B_{i,N_b}(P_{U_{i,N_b[1]}} + P_{U_{i,N_b[2]}} + ... + P_{U_{i,N_b[N_b]}}) \qquad ...[10J.37]$$

But the population figures in brackets in [10J.37] add up to the Australia-wide total population P_{AUS}, as follows:

$$P_{AUS} = P_{U_{i,N_b[1]}} + P_{U_{i,N_b[2]}} + ... + P_{U_{i,N_b[N_b]}} \qquad ...[10J.38]$$

So with [10J.38], [10J.37] simplifies to

$$AE_{AUS} = N_bA_{i,N_b} + B_{i,N_b}P_{AUS} \qquad ...[10J.39]$$

So with [10J.39], [10J.34] becomes:

$$RE_{AUS} = EE_{AUS} – AE_{AUS} = (A_{i,N_b} + B_{i,N_b}P_{AUS}) – (N_bA_{i,N_b} + B_{i,N_b}P_{AUS})$$

which reduces, on cancellation, to:

$$RE_{AUS,LR\text{-}\{s_b,i,N_b\}} = –(N_b–1)A_{i,N_b} = (1–N_b)A_{i,N_b} \qquad ...[10J.40]$$

which is the same as [10J.24] above, as required:

$$RE_{AUS,LR\text{-}\{s_b,i,N_b\}} = (1–N_b)A_{i,N_b} \qquad ...[10J.24]$$

Application of [10J.24] to the LR Technique and the Four PA Technique Estimates

For the case of $N_b = 8$, $\{\mathbf{S}_{b,i,8}\}$ becomes the 8ST regression set, as in [10.60b] in Chapter 10, and [10J.24] reduces to:

$$RE_{AUS,LR-\{\mathbf{S}_{b,i,8}\}} = (1–8)A_{i,8} = –7A_{i,8} \qquad ...[10J.41]$$

Result [10J.41] is the same as [10.33] in Chapter 10 as below, the result which is proven in Appendix 10C:

$$RE_{AUS,LR-8ST} = –7A_{LR} \qquad ...[10.33]$$

The previous sub-section explained how the j = 4 PA technique estimate formula [10.92] (in Chapter 10) includes the ($–4A_{i,5}$) term based on [10J.24] for $N_b = 5$, as in [10J.33]. The j = 3 and j = 2 PA estimate formulas [10.91] and [10.90], respectively, similarly include the ($–3A_{i,4}$) and ($–2A_{i,3}$) terms based on [10J.24] with $N_b = 4$ and $N_b = 3$ in turn. All relative expenditure formulas described by [10J.24] can be explained, as previously in this appendix and also in Appendices 10C and 10F, in terms of multiples of fixed or overhead expenditure (FOE) that are saved in public sector CATs, and which need to be overcome in private sector CATs. For the j = 3 PA technique estimates, for example, for public sector CATs, the ($–3A_{i,4}$) term in [10.91] for $N_b = 4$ describes the three lots of FOE that could be saved if the four STTUs formed after the fourth amalgamation step in sequence i were to horizontally amalgamate into the Australia-wide single STTU (AWSS). Prior to this four into one amalgamation process, Australia-wide expenditures would include four multiples of such FOE, and after amalgamation just a single FOE amount would be incurred, so three lots of FOE, amounting to $3A_{i,4}$ as in [10.91], could be saved.